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Longitudinal dispersion of a buoyant contaminant in a shallow channel

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A nonlinear diffusion equation is derived for the longitudinal dispersion of a buoyant pollutant in a slowly varying current. The essential simplifying feature is that the (non-uniform) water depth is assumed to be much less than the channel width. It is found that for small concentration gradients there is a transverse circulation which leads to a marked reduction in the longitudinal dispersion. For large concentration gradients the longitudinal circulation predominates and the longitudinal dispersion increases.

1. Introduction

The flow of fresh water from rivers into estuaries and the discharge of heated water from industrial plants into rivers or estuaries are two large-scale examples of buoyant contaminants in currents. Such is the magnitude of gravity relative to fluid-mechanical forces that it only needs a small concentration of a buoyant contaminant to have a significant effect upon the flow and hence upon the evolution of the contaminant distribution. Empirically the buoyancy effect can be modelled by permitting the longitudinal dispersion coefficient to depend in some simple way upon the concentration gradient (Harleman & Thatcher 1974; Brocard & Harleman 1976). The aim of this paper is to show how the variation of the dispersion coefficient can be calculated. An analogous calculation for rectangular geometries has recently been given by Chatwin (1976).

For extremely dilute (or neutrally buoyant), laterally well-mixed pollutant distributions in pipe flow Taylor (1953) has shown that the longitudinal evolution is described by a linear diffusion equation. Essentially, this equation describes an asymptotic balance between cross-sectional mixing and longitudinal spreading due to non-uniform advection. The dispersion coefficient (or effective longitudinal diffusivity) depends directly upon the current distribution and inversely upon the lateral diffusivity. Erdogan & Chatwin (1967) point out that for laminar flow there are two effects of buoyancy. First, the longitudinal density gradient causes a longitudinal pressure gradient which changes the velocity distribution and may increase the dispersion. Second, lateral variation of density (due to the non-uniform advection) sets up a secondary flow which augments the cross-sectional mixing and reduces the dispersion. Whether the buoyancy increases or decreases the dispersion depends upon the relative importance of these two effects.

For flow in circular pipes Erdogan & Chatwin (1967) have shown that the leading-order effect of buoyancy is to introduce an extra term in the dispersion coefficient proportional to the square of the concentration gradient. The absence of a linear term is due to the vertical symmetry of circular pipes. Thus we can anticipate that, if the calculation procedure of Erdogan & Chatwin were adapted to apply to open-channel flow, then the leading-order effect of buoyancy would be to make the dispersion coefficient depend linearly upon the concentration gradient. Qualitatively, this agrees with the empirical linear model favoured by Harleman & Thatcher (1974).

In practice, an allowance for buoyancy is needed precisely because it is not a weak effect. Thus the assumptions underlying the calculation procedures of Erdogan & Chatwin are not appropriate. Indeed, it is likely that the empirical linear term for large concentration gradients might differ from that for very small gradients. Instead, the present paper follows Fischer (1967, 1972) and makes explicit use of the fact that rivers and estuaries are typically very shallow relative to their width.

Even with the restriction to vertically well-mixed currents, the wide range and large number of independent parameters precludes us from making a universal model. In the present paper we shall assume that the time scale for current variations is comparable with or longer than the time scale for longitudinal dispersion. This means that the analysis pertains more to rivers than to estuaries. It is hoped to address other cases in subsequent papers.

The major conclusion of the analysis is that the buoyancy can cause a marked reduction in dispersion (i.e. the secondary-flow effect is usually much more important than the longitudinal pressure gradient). This prediction has serious implications for the thermal pollution of rivers. For example, a modest increase in the heat discharged into a river could cause a disproportionately large increase in the maximum temperature.

2. Equations of motion

In seeking a model equation for longitudinal dispersion it is implicit that the pollutant distribution is laterally well mixed. This can be ensured if the length scale of the concentration variations greatly exceeds the channel or pipe width (Taylor 1953). To make this requirement explicit, we introduce a small parameter ϵ, which is the ratio of a typical channel breadth B to the longitudinal length scale L. An immediate implication of the lateral mixing is that to the first approximation the pollutant moves at the cross-sectional average velocity $\overline{U}$. Thus, to study the slow evolution of the pollutant distribution it is convenient to use axes moving at the bulk velocity $\overline{U}$. At this stage in our calculations we shall take $\overline{U}$ to be constant.

Such is the complexity of the situation which we are attempting to model that it is necessary to make many simplifying assumptions. In particular, the channel is taken to be straight and of constant cross-section, turbulent mixing is represented by eddy diffusivities, and we make the Boussinesq approximation (i.e. neglect the inertial but not the buoyancy effect due to density variations).

To achieve further simplification of the mathematical analysis we specify the ϵ-ordering of the many terms in the equations of motion relative to the basic dimensional quantities B and $\bar{U}$. For definiteness we choose the most complicated possibility. This means that for a reasonably wide parameter range no significant terms are neglected although some numerically insignificant terms may be retained. The evolution time scale is of order ϵ^{-2}, the reduced gravity associated with typical density variations is of order ϵ^{-1} and the eddy diffusivities are all of order ϵ^0.

The resulting version of the equations of motion is

$$\epsilon^2\partial_T C+\epsilon(U-\bar{U})\,\partial_X C+V\partial_y C+W\partial_z C$$
$$=\epsilon\,\partial_X(\kappa_1\,\epsilon\,\partial_X C)+\partial_y(\kappa_2\,\partial_y C)+\partial_z(\kappa_3\,\partial_z C), \quad (1a)$$

$$\epsilon^2\partial_T U+\epsilon(U-\bar{U})\,\partial_X U+V\partial_y U+W\partial_z U+\partial_X P$$
$$=\epsilon\,\partial_X(2\nu_{11}\,\epsilon\,\partial_X U)+\partial_y[\nu_{12}(\partial_y U+\epsilon\,\partial_X V)]+\partial_z[\nu_{13}(\partial_z U+\epsilon\,\partial_X W)], \quad (1b)$$

$$\epsilon^2\partial_T V+\epsilon(U-\bar{U})\,\partial_X V+V\partial_y V+W\partial_z V+\epsilon^{-1}\partial_y P$$
$$=\epsilon\,\partial_X[\nu_{12}(\partial_y U+\epsilon\,\partial_X V)]+\partial_y(2\nu_{22}\,\partial_y V)+\partial_z[\nu_{23}(\partial_z V+\partial_y W)], \quad (1c)$$

$$\epsilon^2\partial_T W+\epsilon(U-\bar{U})\,\partial_X W+V\partial_y W+W\partial_z W+\epsilon^{-1}\partial_z P-\epsilon^{-1}\alpha g C$$
$$=\epsilon\,\partial_X[\nu_{13}(\partial_z U+\epsilon\,\partial_X W)]+\partial_y[\nu_{23}(\partial_z V+\partial_y W)]+\partial_z(2\nu_{33}\,\partial_z W), \quad (1d)$$

$$\epsilon\,\partial_X U+\partial_y V+\partial_z W=0, \quad (1e)$$

$$U=V=W=\partial_z C+\partial_y h\,\partial_y C=0 \quad \text{on} \quad z=-h(y), \quad (1f)$$

$$\partial_z U=\partial_z V=W=\partial_z C=0 \quad \text{on} \quad z=\zeta. \quad (1g)$$

Here U, V and W are the velocity components in the downstream, transverse and vertical directions, C is the concentration of the pollutant, $\epsilon^{-1}P$ the excess pressure above mean hydrostatic, $\epsilon^{-1}\alpha g$ the reduced gravity (positive for a buoyant contaminant), the κ_i are eddy diffusivities for salt or heat transport and the ν_{ij} are eddy diffusivities for momentum. Very close to solid surfaces, and everywhere in the special case of laminar flow, we have $\kappa_i=\kappa$ and $\nu_{ij}=\nu$. The simple form of the free-surface boundary condition is a consequence of the Boussinesq approximation (i.e. local fluid-mechanical forces are not adequate to lift the free surface significantly).

We now make explicit use of the fact that ϵ is small and formally seek regular perturbation solutions to (1*a*–*g*) of the form

$$C=C^{(0)}+\epsilon C^{(1)}+\epsilon^2 C^{(2)}+\dots,$$

where the $C^{(j)}$ are all independent of ϵ. For simplicity we shall ignore possible ϵ-dependence of the eddy diffusivities.

The leading-order terms in the diffusion equation (1*a*) are satisfied trivially if $C^{(0)}$ is laterally well mixed. (Indeed, without loss of generality we can identify $C^{(0)}$ with the cross-sectional average concentration $\bar{C}$.) From the lateral momentum equations (1*c*, *d*) it then follows that $P^{(0)}$ is hydrostatic in terms of the density perturbation:

$$P^{(0)}=-\alpha g C^{(0)}(\zeta-z)+\int_0^X(\tfrac{1}{2}\alpha g\,\partial_X C^{(0)}z^{(0)}-F)\,dX.$$

Here $F(X,T)$ is the pressure gradient needed to maintain the constant mass flux, and the $z^{(0)}(X,T)$ term is associated with there being a buoyancy-induced longitudinal circulation. Equation ($1e$) permits us to eliminate the lateral velocities in favour of a stream function:

$$V^{(0)} = \partial_z \Psi, \quad W^{(0)} = -\partial_y \Psi.$$

Using these results in ($1a$–g) we can derive the much simpler equations

$$\partial_z \Psi \partial_y C^{(1)} - \partial_y \Psi \partial_z C^{(1)} - \partial_y(\kappa_2 \partial_y C^{(1)}) - \partial_z(\kappa_3 \partial_z C^{(1)}) = (\bar{U} - U)\partial_X \bar{C}, \qquad (2a)$$

$$\partial_z \Psi \partial_y U^{(0)} - \partial_y \Psi \partial_z U^{(0)} - \partial_y(\nu_{12} \partial_y U^{(0)}) - \partial_z(\nu_{13} \partial_z U^{(0)}) = F + \alpha g[(\zeta - z) - \tfrac{1}{2} z^{(0)}]\partial_X \bar{C}, \qquad (2b)$$

$$\partial_y \Psi \partial_z(\partial_y^2 \Psi + \partial_z^2 \Psi) - \partial_z \Psi \partial_y(\partial_y^2 \Psi + \partial_z^2 \Psi) + (\partial_z^2 - \partial_y^2)[\nu_{23}(\partial_z^2 - \partial_y^2)\Psi] + 2\partial_y \partial_z[(\nu_{22} + \nu_{33})\partial_y \partial_z \Psi] = \alpha g \partial_y C^{(1)}, \qquad (2c)$$

$$U^{(0)} = \Psi = \partial_z \Psi + \partial_y h \partial_y \Psi = \partial_z C^{(1)} + \partial_y h \partial_y C^{(1)} = 0 \quad \text{on} \quad z = -h, \qquad (2d)$$

$$\partial_z U^{(0)} = \Psi = \partial_z^2 \Psi = \partial_z C^{(1)} = 0 \quad \text{on} \quad z = \zeta. \qquad (2e)$$

Integrating the order-ϵ^2 terms in the diffusion equation ($1a$) over the entire cross-section, we find that $\bar{C}$ must satisfy the longitudinal dispersion equation

$$\partial_T \bar{C} + \partial_X(\overline{(U^{(0)} - \bar{U})C^{(1)}}) = \partial_X(\bar{\kappa}_1 \partial_X \bar{C}), \qquad (3)$$

where the overbar denotes the cross-sectional average. To evaluate the shear term (i.e. the term involving $C^{(1)}$ and $U^{(0)}$) we must solve ($2a$–e).

A more painstaking analysis would permit us to determine the parameter range over which (2) and (3) remain essentially unchanged. Specifically, if the eddy diffusivities and reduced gravity are of order ϵ^β and $\epsilon^{\gamma-1}$ then it is necessary that

$$3\beta \leqslant \gamma, \quad \beta > -1 \quad \text{for} \quad \beta \geqslant \gamma, \quad \beta > \tfrac{1}{2}(\gamma - 1) \quad \text{for} \quad \beta \leqslant \gamma$$

(see figure 1). The shear or turbulent diffusion term dominates in (3) according to whether β is positive or negative.

Equations (2) and (3) are a slight generalization of Erdogan & Chatwin's equations (4.7)–(4.9) and (4.15). The present derivation gives a conditional justification of their making the *ad hoc* approximation of retaining longitudinal derivatives of the concentration while neglecting longitudinal derivatives of the buoyancy-driven current. A justification of some of the more important implications, but not of the approximation directly, is given by Barton (1976b).

3. Shallow-water expansion

To make progress we must make further approximations. Erdogan & Chatwin (1967), and more recently Barton (1976a), linearized equations ($2a$–e) by assuming that the buoyancy effect is small. Here we choose to follow Fischer (1967, 1972) and base an approximation scheme on the fact that natural rivers and estuaries are typically much less deep than they are wide. (It is also implicit that the depth is not constant.) Thus we introduce a second small parameter

$$\delta = H/B,$$

where H is a typical channel depth.

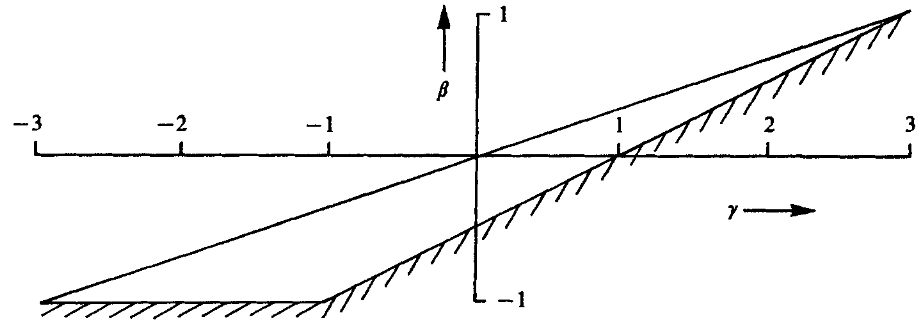

FIGURE 1. Parameter range of the diffusivities, $O(\epsilon^{\beta})$, and of the reduced gravity, $O(\epsilon^{\gamma-1})$, for the applicability of (2) and (3).

We define a new vertical co-ordinate

$$z^* = \delta^{-1}z,$$

and to achieve minimal simplification specify the scalings

$$\nu_{ij} = \delta^{\frac{1}{2}}\nu_{ij}^*, \quad \kappa_i = \delta^{\frac{1}{2}}\kappa_i^*, \quad \alpha = \delta^{-\frac{5}{2}}\alpha^*, \quad C^{(1)} = \delta^{-\frac{1}{2}}C^{(1)*}, \quad \Psi = \delta^{\frac{1}{2}}\Psi^*,$$
$$F = \delta^{-\frac{3}{2}}F^*.$$

Any other choice of scaling for the eddy diffusivities and the reduced gravity simply leads to the neglect of some terms. With the asterisks suppressed the lateral diffusion equation ($2a$) and the corresponding boundary conditions can be written as

$$\left.\begin{aligned} -\partial_z(\kappa_3\,\partial_z C^{(1)}) + \delta\{\partial_z\Psi\,\partial_y C^{(1)} - \partial_y\Psi\,\partial_z C^{(1)}\} - \delta^2\partial_y(\kappa_2\partial_y C^{(1)}) &= \delta^2(\bar{U}-U^{(0)})\,\partial_X\bar{C}, \\ \partial_z C^{(1)} + \delta^2\partial_y h\,\partial_y C^{(1)} = 0 \quad \text{on} \quad z &= -h(y), \\ \partial_z C^{(1)} = 0 \quad \text{on} \quad z &= \zeta. \end{aligned}\right\} \quad (4)$$

At leading order in δ the other equations are

$$-\partial_z(\nu_{13}\,\partial_z U^{(0)}) = F + \alpha g\,\partial_X\bar{C}[(\zeta - z) - \tfrac{1}{2}z^{(0)}],$$

$$\partial_z^2(\nu_{23}\,\partial_z^2\Psi) = \alpha g\,\partial_y C^{(1)},$$

$$U^{(0)} = \Psi = \partial_z\Psi = 0 \quad \text{on} \quad z = -h, \qquad \partial_z U^{(0)} = \Psi = \partial_z^2\Psi = 0 \quad \text{on} \quad z = \zeta.$$

The leading-order solution for the longitudinal velocity is

$$U^{(0)} = F\int_{-h}^{z}\frac{\zeta - z'}{\nu_{13}}\,dz' + \tfrac{1}{2}\alpha g\,\partial_X\bar{C}\int_{-h}^{z}\frac{(\zeta - z')^2 - z^{(0)}(\zeta - z')}{\nu_{13}}\,dz' \qquad (5a)$$

(Prych 1970; Fischer 1972). The functions $F(X,T)$ and $z^{(0)}(X,T)$ are determined such that the $\partial_X\bar{C}$ term makes no net contribution to the volume flux along the channel. In particular, if the eddy viscosity ν_{13} is a function only of the transverse position then we have

$$F = 3\mathscr{A}\bar{U}\Big/\int_{y_-}^{y_+}\frac{(\zeta+h)^3}{\nu_{13}}\,dy, \quad z^{(0)} = \frac{3}{4}\int_{y_-}^{y_+}\frac{(\zeta+h)^4}{\nu_{13}}\,dy\Big/\int_{y_-}^{y_+}\frac{(\zeta+h)^3}{\nu_{13}}\,dy.$$

Here $\mathscr{A}$ is the cross-sectional area and $y_{\pm}$ denote the two sides of the channel. From (5*a*) we note that in the deeper part of the channel (and certainly for $\zeta - z > z^{(0)}$) the buoyancy-driven flow is towards the less dense region. This is in accord with the intuitive expectation that the denser fluid tends to flow under the less dense fluid.

If we use a regular perturbation expansion

$$C^{(1)} = C_0^{(1)} + \delta C_1^{(1)} + \delta^2 C_2^{(1)} + \dots$$

to solve (4) for $C^{(1)}$, then at leading order we find that $C_0^{(1)}$ is vertically well mixed. This fact allows us to solve for Ψ_0:

$$\Psi_0 = \tfrac{1}{2}\alpha g\, \partial_y C_0^{(1)} \int_{-h}^{z} dz' \int_{-h}^{z'} dz'' \left\{\frac{(\zeta - z'')^2 - \psi(\zeta - z'')}{\nu_{23}}\right\}, \tag{5b}$$

where $\psi(y)$ is chosen such that Ψ_0 is zero at the free surface. For vertically constant eddy viscosity we have

$$\psi = \tfrac{3}{4}(\zeta + h).$$

As was the case with the longitudinal circulation, the transverse circulation simply corresponds to the light fluid tending to flow over the denser fluid.

The order-δ terms in (4) are

$$-\partial_z(\kappa_3\, \partial_z C_1^{(1)}) = -\partial_z \Psi_0\, \partial_y C_0^{(1)}, \quad \partial_z C_1^{(1)} = 0 \quad \text{on} \quad z = -h, \zeta.$$

Without loss of generality we take the solution to be

$$C_1^{(1)} = \tfrac{1}{2}\alpha g(\partial_y C_0^{(1)})^2 \int_{-h}^{z} \frac{dz'}{\kappa_3} \int_{-h}^{z'} dz'' \int_{-h}^{z''} dz''' \left\{\frac{(\zeta - z''')^2 - \psi(\zeta - z''')}{\nu_{23}}\right\}. \tag{5c}$$

Thus $C_1^{(1)}$ describes the slight vertical stratification caused by the transverse flow. For vertically constant eddy diffusivities the density difference between the channel bed and the free surface is

$$\alpha^2 g(\partial_y C_0^{(1)})^2 (h + \zeta)^5 / 320 \kappa_3 \nu_{23}.$$

Integrating the order-δ^2 terms in the diffusion equation (4) between the channel bed and the free surface, we find that $C_0^{(1)}$ must satisfy the transverse dispersion equation

$$\partial_y \left(\int_{-h}^{\zeta} \kappa_2\, dz\, \partial_y C_0^{(1)} \right) + \partial_y \Bigg(\tfrac{1}{4}\alpha^2 g^2 (\partial_y C_0^{(1)})^3 \times \int_{-h}^{\zeta} \frac{dz}{\kappa_3} \left[\int_{-h}^{z} dz' \int_{-h}^{z'} dz'' \left\{\frac{(\zeta - z'')^2 - \psi(\zeta - z'')}{\nu_{23}}\right\} \right]^2 \Bigg) = \partial_X \bar{C} \int_{-h}^{\zeta} (U^{(0)} - \bar{U})\, dz.$$

At the sides of the channel we have the zero-flux condition

$$(\zeta + h)\, \partial_y C_0^{(1)} = 0.$$

One integration with respect to y yields a cubic equation for $\partial_y C_0^{(1)}$ which, making use of (5*a*), we can write as

$$\partial_y C_0^{(1)} \{K_0(y) + (\partial_y C_0^{(1)})^2 K_2(y)\} = \partial_X \bar{C} \{Q_0(y) + \partial_X \bar{C} Q_1(y)\}, \tag{5d}$$

where we have used the abbreviated notation

$$K_0 = \int_{-h}^{\zeta} \kappa_2\, dz, \quad K_2 = \tfrac{1}{4}\alpha^2 g^2 \int_{-h}^{\zeta} \frac{dz}{\kappa_3}\left[\int_{-h}^{z} dz' \int_{-h}^{z'} dz'' \left\{\frac{(\zeta - z'')^2 - \psi(\zeta - z'')}{\nu_{23}}\right\}\right]^2,$$

$$Q_0 = \int_{y-}^{y} dy' \int_{-h}^{\zeta} dz \left[F\int_{-h}^{z} \frac{\zeta - z'}{\nu_{13}}\, dz' - \overline{U}\right],$$

$$Q_1 = \tfrac{1}{2}\alpha g \int_{y-}^{y} dy' \int_{-h}^{\zeta} dz \int_{-h}^{z} dz' \left\{\frac{(\zeta - z')^2 - z^{(0)}(\zeta - z')}{\nu_{13}}\right\}.$$

The positivity of the coefficients on the left-hand side of (5*d*) ensures that there is a unique solution for $\partial_y C_0^{(1)}$.

For subsequent identification we note that K_0 is associated with the turbulent transverse mixing, K_2 with the buoyancy-induced transverse shear dispersion, Q_0 with the shear flow, and Q_1 with the buoyancy-induced longitudinal circulation. For vertically constant eddy diffusivities we have

$$K_0 = \kappa_2(\zeta + h), \quad K_2 = \frac{19(\zeta + h)^9 \alpha^2 g^2}{7.10.12^4\, \nu_{23}^2 \kappa_3},$$

$$Q_0 = \tfrac{1}{3}F \int_{y-}^{y} \frac{(\zeta + h)^3}{\nu_{13}}\, dy' - \overline{U}\int_{y-}^{y} (\zeta + h)\, dy'.$$

$$Q_1 = \tfrac{1}{8}\alpha g \left\{\int_{y-}^{y} \frac{(\zeta + h)^4}{\nu_{13}}\, dy' - \tfrac{4}{3} z^{(0)} \int_{y-}^{y} \frac{(\zeta + h)^3}{\nu_{13}}\, dy'\right\}.$$

4. Dispersion coefficient

Although the use of moving axes was appropriate in the formal derivation of the results (3) and (5), in applications it is more convenient to use stationary axes. Similarly, it may be desirable to re-interpret the equations in terms of dimensional variables or in terms of some scaling other than the minimal-simplification scaling. Of course, the validity of the equations still depends upon the validity of the underlying assumptions (i.e. that the pollutant distribution is laterally well mixed and that the channel is shallow). In (5*a–d*) we merely need to replace ∂_X by ∂_x, and we replace the longitudinal dispersion (3) by

$$\partial_t(\mathscr{A}\overline{C}) + \partial_x(\mathscr{A}\overline{U}\overline{C}) + \partial_x\left(\int_{y-}^{y+} dy\, C_0^{(1)} \int_{-h}^{\zeta} dz (U^{(0)} - \overline{U})\right) = \partial_x(\bar{\kappa}_1 \mathscr{A}\partial_x \overline{C}).$$

The occurrence of the cross-sectional area $\mathscr{A}$ under the differential operators allows for the possibility of very gradual depth or profile changes (Harleman & Thatcher 1974). A direct derivation including this effect shows that the time scale for current variations should be comparable with or longer than the time scale for longitudinal dispersion. An integration by parts enables us to rewrite the shear diffusion term:

$$\partial_t(\mathscr{A}\overline{C}) + \partial_x(\mathscr{A}\overline{U}\overline{C}) = \partial_x[(\bar{\kappa}_1 + E)\mathscr{A}\partial_x \overline{C}], \tag{6a}$$

with

$$E = \frac{1}{\mathscr{A}}\int_{y-}^{y+} dy \frac{\partial_y C_0^{(1)}}{\partial_x \overline{C}} \int_{y-}^{y} dy' \int_{-h}^{\zeta} dz (U^{(0)} - \overline{U}) = \frac{1}{\mathscr{A}}\int_{y-}^{y+} dy \frac{\{Q_0(y) + \partial_x \overline{C} Q_1(y)\}^2}{K_0(y) + (\partial_y C_0^{(1)})^2 K_2(y)}. \tag{6b}$$

As befits a diffusivity, the shear dispersion coefficient E is necessarily positive.

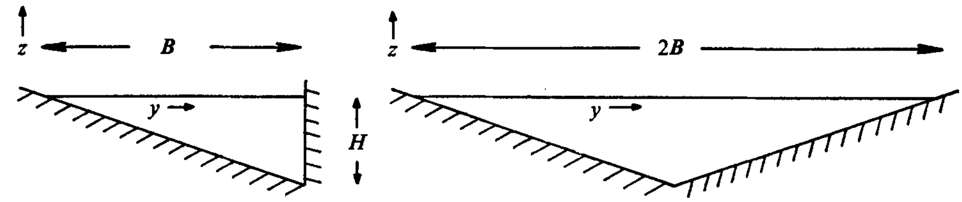

FIGURE 2. Definition sketch for the cross-section used in the analysis.

For weak buoyancy effects it is reasonable to model the eddy diffusivities as being independent of the concentration gradient $\partial_x \bar{C}$. This leads to E having the Taylor-series expansion

$$E = \frac{1}{\mathscr{A}} \int_{y-}^{y+} dy \left[\frac{Q_0^2}{K_0} + 2\partial_x \bar{C} \frac{Q_0 Q_1}{K_0} + (\partial_x \bar{C})^2 \left\{ \frac{Q_1^2}{K_0} - \frac{Q_0^4 K_2}{K_0^4} \right\} - 4(\partial_x \bar{C})^3 \frac{Q_0^3 Q_1 K_2}{K_0^4} + \dots \right]. \tag{7}$$

This formula makes quite clear the order in which the various circulation mechanisms come into play as the buoyancy effect increases. First there is shear dispersion augmented by longitudinal turbulent mixing (Taylor 1953; Fischer 1967). Next, the longitudinal circulation becomes important (Prych 1970; Fischer 1972). Finally, the transverse circulation comes into play and tends to reduce the dispersion (Erdogan & Chatwin 1967; Barton 1976*b*). These qualitative considerations remain valid for more complicated models in which the eddy diffusivities depend upon $\partial_x \bar{C}$.

In the absence of buoyancy, a reasonable approximation to the local eddy diffusivities can be obtained from the formulae

$$\kappa_i = \kappa_i' |\hat{U}| (h + \zeta), \quad \nu_{ij} = \nu_{ij}' |\hat{U}| (h + \zeta), \tag{8}$$

where $\hat{U}$ is the velocity at the free surface and ν_{ij}' and κ_i' are numerical constants. These formulae presume a linear relationship between the shear velocity U^* and the more readily measured (or calculated) free-surface velocity. Typically, in field conditions the vertical transport coefficients κ_3', ν_{13}' and ν_{23}' are of order 0·005 and the horizontal transport coefficients κ_1' and κ_2' are of order 0·02. The above formulae are equivalent to modelling the eddy diffusivities as being proportional to $(h+\zeta)^{\frac{3}{2}}$. Fischer (1972) proposed a similar model in which the diffusivities depend linearly upon the local depth.

For a channel of triangular cross-section (as shown in figure 2) we find

$$K_0 = \tfrac{15}{8} \kappa_2' |\bar{U}| H^2 (y/B)^{\frac{5}{2}}, \quad K_2 = \frac{19 H^6 \alpha^2 g^2}{7 \, . \, 5^4 \, . \, 3^7 |\bar{U}|^3 (\nu_{23}')^2 \kappa_3'} (y/B)^{\frac{9}{2}},$$

$$Q_0 = \tfrac{1}{2} \bar{U} H B \{ (y/B)^{\frac{5}{2}} - (y/B)^2 \}, \quad Q_1 = \frac{2\alpha g B H^3}{3 \, . \, 5 \, . \, 7 |\bar{U}| \nu_{13}'} \{ (y/B)^{\frac{7}{2}} - (y/B)^{\frac{5}{2}} \}.$$

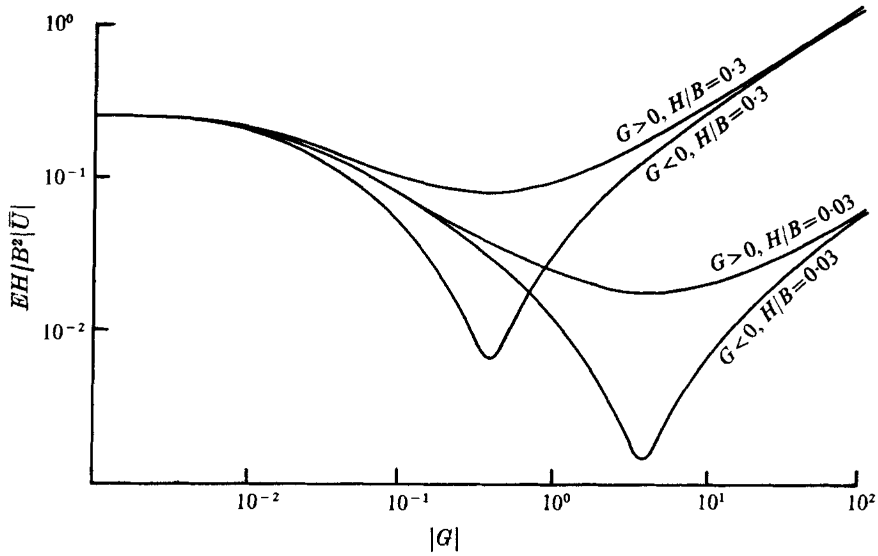

FIGURE 3. Longitudinal dispersion coefficient as a function of the local concentration gradient.

Assuming that these expressions remain valid for weak buoyancy, we can determine the coefficients in the Taylor series (7) for E explicitly:

$$E = \frac{B^2|\bar{U}|}{H}\left[\frac{2^3}{3^2.5^2.7\kappa_2'} + G\left(\frac{H}{B}\right)\frac{2^3}{3^2.5.7^2\kappa_2'\,\nu_{13}'} + G^2\left(\frac{H}{B}\right)^2\frac{2^8}{3^4.5^3.7^3.11\kappa_2'(\nu_{13}')^2}\right.$$
$$\left. - G^2\frac{2^{10}.19}{3^{12}.5^9.7.11\kappa_3'(\nu_{23}')^2(\kappa_2')^4} - G^3\left(\frac{H}{B}\right)\frac{2^{11}.19}{3^{13}.5^{10}.7.11.13\kappa_3'\,\nu_{13}'(\nu_{23}')^2(\kappa_2')^4} + \dots\right],$$

where G is the local longitudinal gradient Richardson number, i.e.

$$G = \alpha g H B\,\partial_x \bar{C}/\bar{U}|\bar{U}|. \tag{9}$$

G positive corresponds to the fluid being less dense downstream. The repeated occurrence of ν_{ij}' and κ_j' in the denominators permits the coefficients to be quite large. Specifically, with the horizontal and vertical transport coefficients set equal to 0·02 and 0·005 respectively, we have

$$E = \frac{B^2|\bar{U}|}{H}\left[0{\cdot}254 + 36{\cdot}2G\left(\frac{H}{B}\right) + 13{\cdot}4G^2\left(\frac{H}{B}\right)^2 - 1736G^2 - 24966G^3\left(\frac{H}{B}\right) + \dots\right].$$

Thus we infer that even for very small G the transverse circulation can cause a marked reduction in the longitudinal dispersion (see figure 3). For channels of other shapes only the numerical factors are changed.

For large buoyancy effects the modelling of the eddy diffusivities is more crucial. First, the lateral distribution of the current and hence of the eddy diffusivities is changed. Second, the vertical stratification tends to reduce the turbulent transport (Turner 1973, p. 161). For rectangular channels only the latter effect is operative and there is an increase in the time needed to achieve vertical mixing and an increase in the longitudinal dispersion (Ippen & Harleman

1961). Fortunately, for non-rectangular channels the buoyancy-induced motion is primarily horizontal, and the vertical stratification $C_1^{(1)}$ is two orders smaller than the longitudinal stratification. Thus we can hope that there is a significant range of the longitudinal gradient Richardson number G in which the vertical stratification plays a negligible role. This can be checked using the solution (5*c*) for $C_1^{(1)}$.

The simplest possibility is to ignore any dependence of the eddy diffusivities upon the concentration gradient $\partial_x \bar{C}$. This leads to the asymptotic representation

$$E \sim \frac{1}{\mathscr{A}} \int_{y-}^{y+} dy\,[(\partial_x \bar{C})^{\frac{2}{3}} Q_1^{\frac{4}{3}} K_2^{-\frac{1}{3}} + \tfrac{4}{3}(\partial_x \bar{C})^{-\frac{1}{3}} Q_0 Q_1^{\frac{1}{3}} K_2^{-\frac{1}{3}} + \ldots].$$

Thus, for sufficiently large concentration gradients the longitudinal circulation causes an increase in the longitudinal dispersion (see figure 3).

A more realistic way of modelling the eddy-diffusivity distribution would be to make direct use of the empirical relationship (8). This leads to a considerable increase in the complexity of the analysis. For example (5*a*) becomes

$$\nu_{13}' \hat{U} |\hat{U}| = \tfrac{1}{2} F(\zeta + h) + \tfrac{1}{6} \alpha g \partial_x \bar{C} \{(\zeta + h)^2 - \tfrac{3}{2} z^{(0)} (\zeta + h)\},$$

where F and $z^{(0)}$ are nonlinear functionals of $\hat{U}$. The solutions will be very much as before until G becomes of order $\nu_{13}' B/H$. Thereafter the large longitudinal circulation causes increased turbulent transport and slightly inhibits the increase in the longitudinal dispersion. For large $\partial_x \bar{C}$ it can be ascertained that E grows as $|\partial_x \bar{C}|^{\frac{1}{2}}$.

Both the above models predict smaller growth rates of the dispersion than does the empirical linear model of Harleman & Thatcher (1974). This is compatible with the fact that here we have ignored the role of vertical stratification whereas the range of observations used by Harleman & Thatcher extends to relatively stratified conditions.

5. Practical implications

The density differences associated with salinity are several hundred times larger than those associated with temperature. Thus we can anticipate that strong buoyancy effects will be apparent in estuary flows. There is no such thing as a typical estuary, but to permit quantitative statements we specify the parameter values

$$\bar{U} \sim 0{\cdot}5\,\mathrm{m\,s^{-1}}, \quad B \sim 150\,\mathrm{m}, \quad H \sim 20\,\mathrm{m}.$$

An essential requirement of the above analysis is that the flow is laterally well mixed. An estimate of the cross-sectional mixing time is

$$B^2/8\bar{\kappa}_2 \sim 4\,\mathrm{h}.$$

This is a considerable overestimate if there is any transverse circulation. If we take C to measure the salinity in parts per thousand and measure longitudinal distance in kilometres, then we get

$$\alpha g \sim -7{\cdot}5 \times 10^{-3}\,\mathrm{m\,s^{-2}}, \quad G \sim -9 \times 10^{-2} \partial_x \bar{C}.$$

The ocean salinity is typically 32 parts per thousand. Thus, using figure 3, we infer that the dispersion is strongly affected by buoyancy unless the mixing between fresh and salt water extends over 100 km.

As the tide turns the value of G increases without bound, and according to the present scalings E (but not $\bar{C}$) has a weak singularity. A more appropriate scaling for the turning of the tide would show that E briefly becomes extremely large but finite. For most of the tide the dispersion will be of order $1\,\mathrm{km^2\,day^{-1}}$. If we were strictly to abide by the quasi-stationarity assumption, then for the dispersion time scale to be of the order of one day it would be necessary for the mixing region to have a length of only several kilometres. By virtue of the minimal-simplification structure of the above calculation procedure, it is plausible that (6 *a*, *b*) remain applicable when the dispersion is only a minor (but systematic) addition to the back-and-forth advection of the fluid. In a subsequent paper it is hoped to present an alternative calculation procedure which is more purpose-built for estuaries.

For a navigable river a possible specification of river conditions is

$$\bar{U} \sim 0{\cdot}1\,\mathrm{m\,s^{-1}}, \quad B \sim 20\,\mathrm{m}, \quad H \sim 5\,\mathrm{m}.$$

Cross-sectional mixing would take about one and a half hours, or equivalently would be achieved within half a kilometre of the source of the contaminant. If we take heat to be the source of buoyancy, measure C in degrees Celsius and measure longitudinal distance in kilometres then we get

$$\alpha g \sim 10^{-3}\,\mathrm{m\,s^{-2}}, \quad G \sim 10^{-2}\,\partial_x \bar{C}.$$

Thus, for buoyancy significantly to inhibit the dispersion (and to accelerate the cross-sectional mixing) the temperature gradient would have to exceed $2\,°\mathrm{C\,km^{-1}}$. This would correspond to an industrial plant taking much less than two hours to build up to or close down from a 50 MW cooling requirement. For small concentration gradients the dispersion coefficient E is of order $0{\cdot}17\,\mathrm{km^2\,day^{-1}}$. This means that a 2 km long plug of heated water is effectively dispersed in one day. However, a rapid build-up and close-down seals off the ends of the plug (by means of reducing the dispersion) and permits it to preserve its identity for several days. If this were repeated at industrial plants further down the river then unexpectedly large temperatures could arise. Similar considerations apply to processes in which very hot water is discharged intermittently into a river. An increase in the amount of heat discharged would reduce the dispersion and would cause a greater temperature increase than might be anticipated.

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